

TWISTING FORMULA OF EPSILON FACTORS

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ABSTRACT. For characters of a nonarchimedean local field we have explicit formula for epsilon factor. But in general we do not have any generalized twisting formula of epsilon factors. In this paper we give a twisting formula of epsilon factors via local Jacobi sums.

1. Introduction

Let F be a non-achimedean local field and χ be a character of F^\times . Let ψ be an additive character of F . By Langlands we can associate a local constant (or local epsilon factor) with each character of local field. This epsilon factor appears in the functional equation of Tate (cf. [7]). Tate [6] gives an explicit formula for $\epsilon(\chi, \psi)$. But in general we do not have any explicit formula of epsilon factor of a character twisted by a character. Let χ_1 and χ_2 be two characters of F^\times . If any one of these character is unramified, then we have a formula for $\epsilon(\chi_1\chi_2, \psi)$ due to Tate. Also if conductor, $a(\chi_1) \geq 2a(\chi_2)$, then by Deligne (cf. [8]) we have a formula for $\epsilon(\chi_1\chi_2, \psi)$. In this article we give general twisting formula for $\epsilon(\chi_1\chi_2, \psi)$, when both χ_1 and χ_2 are ramified.

Let U_F be the group of units in O_F (ring of integers of F) and π_F be a uniformizer of F . Let χ be a character of F^\times with conductor $a(\chi)$. Let ψ be additive character of F with conductor $n(\psi)$. Then the epsilon factor of character χ is (cf. equation (3.13)):

$$(1.1) \quad \epsilon(\chi, \psi) = \chi(c)q^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(x/c),$$

where $c = \pi_F^{a(\chi)+n(\psi)}$, and q is the cardinality of the residue field F .

For characters χ_1, χ_2 of F^\times and a positive integer n , we define the local Jacobi sum:

$$(1.2) \quad J_t(\chi_1, \chi_2, n) = \sum_{\substack{x \in \frac{U_F}{U_F^n} \\ t-x \in U_F}} \chi_1^{-1}(x)\chi_2^{-1}(t-x).$$

In Theorem 4.5, we give the twisting formula of epsilon factor via local Jacobi sums. In our twisting formula conductors play important role and the formula is:

$$(1.3) \quad \epsilon(\chi_1\chi_2, \psi) = \begin{cases} \frac{q^{\frac{r}{2}} \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = m = r, \\ \frac{q^{\frac{r}{2}} \chi_1\chi_2(\pi_F^{r-n}) \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = m > r, \\ \frac{q^{n-\frac{m}{2}} \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = r > m, \end{cases}$$

where $n = a(\chi_1)$, $m = a(\chi_2)$ and $r = a(\chi_1\chi_2)$.

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2. Notations

Let F be a non-archimedean local field, i.e., a finite extension of the field \mathbb{Q}_p (field of p -adic numbers), where p is a prime. Let O_F be the ring of integers in local field F and $P_F = \pi_F O_F$ is the unique prime ideal in O_F and π_F is a uniformizer, i.e., an element in P_F whose valuation is one i.e., $\nu_F(\pi_F) = 1$. The cardinality of the residue field of F is q , i.e., $|O_F/P_F| = q$. Let $U_F = O_F - P_F$ be the group of units in O_F . Let $P_F^i = \{x \in F : \nu_F(x) \geq i\}$ and for $i \geq 0$ define $U_F^i = 1 + P_F^i$ (with proviso $U_F^0 = U_F = O_F^\times$).

Definition 2.1 (Canonical additive character). We define the non trivial additive character of F , $\psi_F : F \rightarrow \mathbb{C}^\times$ as the composition of the following four maps:

$$F \xrightarrow{\text{Tr}_{F/\mathbb{Q}_p}} \mathbb{Q}_p \xrightarrow{\alpha} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\beta} \mathbb{Q}/\mathbb{Z} \xrightarrow{\gamma} \mathbb{C}^\times,$$

where

- (1) $\text{Tr}_{F/\mathbb{Q}_p}$ is the trace from F to \mathbb{Q}_p ,
- (2) α is the canonical surjection map,
- (3) β is the canonical injection which maps $\mathbb{Q}_p/\mathbb{Z}_p$ onto the p -component of the divisible group \mathbb{Q}/\mathbb{Z} and
- (4) γ is the exponential map $x \mapsto e^{2\pi i x}$, where $i = \sqrt{-1}$.

For every $x \in \mathbb{Q}_p$, there is a rational r , uniquely determined modulo 1, such that $x - r \in \mathbb{Z}_p$. Then $\psi_{\mathbb{Q}_p}(x) = \psi_{\mathbb{Q}_p}(r) = e^{2\pi i r}$. The nontrivial additive character $\psi_F = \psi_{\mathbb{Q}_p} \circ \text{Tr}_{F/\mathbb{Q}_p}$ of F is called the **canonical additive character** (cf. [5], p.92).

Definition 2.2 (Conductor of characters). The conductor of any nontrivial additive character ψ of a field is an integer $n(\psi)$ if ψ is trivial on $P_F^{-n(\psi)}$, but nontrivial on $P_F^{-n(\psi)-1}$. We also consider $a(\chi)$ as the conductor of nontrivial character $\chi : F^\times \rightarrow \mathbb{C}^\times$, i.e., $a(\chi)$ is the smallest integer $m \geq 0$ such that χ is trivial on U_F^m . We say χ is unramified if the conductor of χ is zero and otherwise ramified. We also recall here that for two characters χ_1 and χ_2 of F^\times we have $a(\chi_1 \chi_2) \leq \max(a(\chi_1), a(\chi_2))$ with equality if $a(\chi_1) \neq a(\chi_2)$.

3. Epsilon factors

Let F be a nonarchimedean local field and χ be a character of F^\times . The $L(\chi)$ -functions are defined as follows:

$$L(\chi) = \begin{cases} (1 - \chi(\pi_F))^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified.} \end{cases}$$

We denote by dx a Haar measure on F , by $d^\times x$ a Haar measure on F^\times and the relation between these two Haar measure is:

$$d^\times x = \frac{dx}{|x|},$$

for arbitrary Haar measure dx on F . For given additive character ψ of F and Haar measure dx on F , we have a **Fourier transform** as:

$$(3.1) \quad \hat{f}(y) = \int f(x) \psi(xy) dx.$$

where $f \in L^1(F^+)$ (that is, $|f|$ is integrable) and the Haar measure is normalized such that $\hat{f}(y) = f(-y)$, i.e., dx is self-dual with respect to ψ . By Tate (cf. [6], p.13), for any character χ of F^\times , there exists a number $\epsilon(\chi, \psi, dx) \in \mathbb{C}^\times$ such that it satisfies the following local functional equation:

$$(3.2) \quad \frac{\int \hat{f}(x) w_1 \chi^{-1}(x) d^\times x}{L(w_1 \chi^{-1})} = \epsilon(\chi, \psi, dx) \frac{\int f(x) \chi(x) d^\times x}{L(\chi)}.$$

for any such function f for which the both sides make sense. This number $\epsilon(\chi, \psi, dx)$ is called the **local epsilon factor or local constant** of χ .

For a nontrivial multiplicative character χ of F^\times and non-trivial additive character ψ of F , we have (cf. [9], p.4)

$$(3.3) \quad \epsilon(\chi, \psi, c) = \chi(c) \frac{\int_{U_F} \chi^{-1}(x) \psi(x/c) dx}{\left| \int_{U_F} \chi^{-1}(x) \psi(x/c) dx \right|}$$

where the Haar measure dx is normalized such that the measure of O_F is 1 and where $c \in F^\times$ with valuation $n(\psi) + a(\chi)$. To get modified **summation formula** of epsilon factor from the integral formula (3.3), we need the next lemma which can be found in [2], but here we give different proof.

Lemma 3.1. *Let χ be a nontrivial character of F^\times with conductor $a(\chi)$. Let ψ be an additive character of F with conductor $n(\psi)$. We define the integration for an integer $m \in \mathbb{Z}$:*

$$(3.4) \quad I(m) = \int_{U_F} \chi^{-1}(x) \psi\left(\frac{x}{\pi_F^{l+m}}\right) dx, \quad \text{where } l = a(\chi) + n(\psi).$$

Then

$$(3.5) \quad |I(m)| = \begin{cases} q^{-\frac{a(\chi)}{2}} & \text{when } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We divide this proof into three cases.

Case-1 when $m = 0$. In general we can write

$$\begin{aligned} I(m) &= \int_{U_F} \chi^{-1}(x) \psi\left(\frac{x}{\pi_F^{l+m}}\right) dx \\ &= \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{x}{\pi_F^{l+m}}\right) \times m'(U_F^{a(\chi)}) \\ &= q^{-a(\chi)} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{x}{\pi_F^{l+m}}\right). \end{aligned}$$

In the above equation m' is the normalized Haar measure of F so that $m'(O_F) = 1$. Also $O_F = \cup_{a_i \in \mathbb{F}_q} (a_i + \pi_F U_F) : n = 1, 2, \dots$. Again $|\mathbb{F}_q| = q$ and each element of this union has measure $1/q$, then $m'(U_F) = 1 - 1/q = \frac{q-1}{q}$. Since $U_F = \frac{U_F}{U_F^{a(\chi)}} U_F^{a(\chi)}$, this implies $m'(U_F^{a(\chi)}) = m'(U_F) / \left| \frac{U_F}{U_F^{a(\chi)}} \right|$. we have $\left| \frac{U_F}{U_F^{a(\chi)}} \right| = (q-1)q^{a(\chi)-1}$, therefore $m'(U_F^{a(\chi)}) = q^{-a(\chi)}$.

Now putting $m = 0$ we have

$$(3.6) \quad I(0) = q^{-a(\chi)} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{x}{\pi_F^l}\right),$$

Therefore we have

$$(3.7) \quad \overline{I(0)} = q^{-a(\chi)} \sum_{y \in \frac{U_F}{U_F^{a(\chi)}}} \chi(y) \psi\left(-\frac{y}{\pi_F^l}\right),$$

since $\bar{\chi}(x) = \chi^{-1}(x)$ and $\bar{\psi}(x) = \psi(-x)$ for any $x \in F^\times$ and $a(\bar{\chi}) = a(\chi)$. We have

$$(3.8) \quad \begin{aligned} I(0)\overline{I(0)} &= q^{-2a(\chi)} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{x}{\pi_F^l}\right) \times \sum_{y \in \frac{U_F}{U_F^{a(\chi)}}} \chi(y) \psi\left(-\frac{y}{\pi_F^l}\right) \\ &= q^{-2a(\chi)} \sum_{x, y \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \chi(y) \psi\left(\frac{x-y}{\pi_F^l}\right) \\ &= q^{-2a(\chi)} \sum_{x, y \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{xy-y}{\pi_F^l}\right) \quad \text{replacing } x \text{ by } xy \\ (3.9) \quad &= q^{-2a(\chi)} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \varphi(x). \end{aligned}$$

where

$$(3.10) \quad \varphi(x) = \sum_{y \in \frac{U_F}{U_F^{a(\chi)}}} \psi\left(y \frac{x-1}{\pi_F^l}\right).$$

Since $\frac{U_F}{U_F^{a(\chi)}} = \left(\frac{O_F}{P_F^{a(\chi)}}\right)^\times = \frac{O_F}{P_F^{a(\chi)}} \setminus \frac{P_F}{P_F^{a(\chi)}}$, therefore $\varphi(x)$ can be written as the difference

$$\begin{aligned} \varphi(x) &= \sum_{y \in \frac{U_F}{U_F^{a(\chi)}}} \psi\left(y \frac{x-1}{\pi_F^l}\right) \\ &= \sum_{y \in \frac{O_F}{P_F^{a(\chi)}}} \psi\left(y \frac{x-1}{\pi_F^l}\right) - \sum_{y \in \frac{P_F}{P_F^{a(\chi)}}} \psi\left(y \frac{x-1}{\pi_F^l}\right) \\ &= \sum_{y \in \frac{O_F}{P_F^{a(\chi)}}} \psi\left(y \frac{x-1}{\pi_F^l}\right) - \sum_{y \in \frac{O_F}{P_F^{a(\chi)-1}}} \psi\left(y \frac{(x-1)\pi_F}{\pi_F^l}\right) \\ &= A - B, \end{aligned}$$

where $A = \sum_{y \in \frac{O_F}{P_F^{a(\chi)}}} \psi(y \frac{x-1}{\pi_F^l})$ and $B = \sum_{y \in \frac{O_F}{P_F^{a(\chi)-1}}} \psi(y \frac{(x-1)\pi_F}{\pi_F^l})$. It is easy to see (cf. [4], p.28, Lemma 2.1) that

$$\sum_{y \in \frac{O_F}{P_F^{a(\chi)}}} \psi(y\alpha) = \begin{cases} q^{a(\chi)} & \text{when } \alpha \in P_F^{-n(\psi)} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $A = q^{a(\chi)}$ when $x \in U_F^{a(\chi)}$ and $A = 0$ otherwise. Similarly $B = q^{a(\chi)-1}$ when $x \in U_F^{a(\chi)-1}$ and $B = 0$ otherwise. Therefore we obtain

$$(3.11) \quad \varphi(x) = \sum_{y \in \frac{U_F}{U_F^{a(\chi)}}} \psi(y \frac{x-1}{\pi_F^l}) = q^{a(\chi)} - q^{a(\chi)-1} \cdot \sum_{x \in \frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}} \chi^{-1}(x).$$

Moreover, since the conductor of χ is $a(\chi)$, then it can be proved that $\sum_{x \in \frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}} \chi^{-1}(x) = 0$. So we have

$$\begin{aligned} I(0) \cdot \overline{I(0)} &= |I(0)|^2 \\ &= q^{-a(\chi)} - q^{a(\chi)-1} \sum_{x \in \frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}} \chi^{-1}(x) \\ &= q^{-a(\chi)}. \end{aligned}$$

Hence it proves for $m = 0$,

$$|I(m)| = q^{-\frac{a(\chi)}{2}}.$$

Case-2 when $m < 0$. In this case

$$\begin{aligned} I(m) &= \int_{U_F} \chi^{-1}(x) \psi(\frac{x}{\pi_F^{l+m}}) dx \\ &= \int_{\frac{U_F}{U_F^r}} \chi^{-1}(x) \left(\int_{U_F^r} \chi^{-1}(y) \psi(\frac{xy}{\pi_F^{l+m}}) dy \right) dx. \end{aligned}$$

Let $I'(m)$ be the inner integral of the above equation. Then we have

$$\begin{aligned} I'(m) &= \int_{U_F^r} \chi^{-1}(y) \psi(\frac{xy}{\pi_F^{l+m}}) dy \\ &= \int_{P_F^r} \chi^{-1}(1+z) \psi(\frac{x(1+z)}{\pi_F^{l+m}}) dz \quad \text{replacing } y \text{ by } 1+z \\ &= \psi(\frac{x}{\pi_F^{l+m}}) \int_{P_F^r} \chi^{-1}(1+z) \psi(\frac{xz}{\pi_F^{l+m}}) dz. \end{aligned}$$

Let $r = a(\chi) + m$, then the valuation of $\frac{xz}{\pi_F^{l+m}}$ is $\nu_F(\frac{xz}{\pi_F^{l+m}}) = a(\chi) + m - a(\chi) - n(\psi) - m = -n(\psi)$, therefore $\psi(\frac{xz}{\pi_F^{l+m}}) = 1$ for all $z \in P_F^{a(\chi)+m}$. This implies the integral

$$\begin{aligned}
 I'(m) &= \psi\left(\frac{x}{\pi_F^{l+m}}\right) \int_{P_F^{a(\chi)+m}} \chi^{-1}(1+z) \psi\left(\frac{xz}{\pi_F^{l+m}}\right) dz \\
 &= \psi\left(\frac{x}{\pi_F^{l+m}}\right) \int_{P_F^{a(\chi)+m}} \chi^{-1}(1+z) dz \\
 &= \psi\left(\frac{x}{\pi_F^{l+m}}\right) \int_{U_F^{a(\chi)+m}} \chi^{-1}(y) dy \\
 &= \psi\left(\frac{x}{\pi_F^{l+m}}\right) \times 0 \\
 &= 0.
 \end{aligned}$$

Since $\chi^{-1}(y) \neq 1$ over $U_F^{a(\chi)+m}$ ($m < 0$). Therefore in this case we have

$$I(m) = \int_{\frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) dx \times 0 = 0.$$

Case-3 when $m > 0$. In this case we can write

$$\begin{aligned}
 I(m) &= \int_{U_F} \chi^{-1}(x) \psi\left(\frac{x}{\pi_F^{l+m}}\right) dx \\
 &= \int_{\frac{U_F}{U_F^{a(\chi)+m-r}}} \chi^{-1}(x) \left(\int_{U_F^{a(\chi)+m-r}} \chi^{-1}(y) \psi\left(\frac{xy}{\pi_F^{l+m}}\right) dy \right) dx,
 \end{aligned}$$

where $1 \leq r \leq m$.

Let $I''(m)$ be the inner integral of the above equation and we have

$$\begin{aligned}
 I''(m) &= \int_{U_F^{a(\chi)+m-r}} \chi^{-1}(y) \psi\left(\frac{xy}{\pi_F^{l+m}}\right) dy \\
 &= \int_{U_F^{a(\chi)+m-r}} \psi\left(\frac{xy}{\pi_F^{l+m}}\right) dy \quad \text{since conductor of } \chi \text{ is } a(\chi) \\
 &= \int_{P_F^{a(\chi)+m-r}} \psi\left(\frac{x(1+z)}{\pi_F^{l+m}}\right) dz \\
 &= \psi\left(\frac{x}{\pi_F^{l+m}}\right) \int_{P_F^{a(\chi)+m-r}} \psi\left(\frac{xz}{\pi_F^{l+m}}\right) dz \\
 &= \psi\left(\frac{x}{\pi_F^{l+m}}\right) \times 0 \\
 &= 0,
 \end{aligned}$$

since $\psi(\frac{xz}{\pi_F^{l+m}}) \neq 1$ for $z \in P_F^{a(\chi)+m-r}$. Therefore $I(m) = 0$. This completes the proof. \square

Hence the formula (3.3) reduces to

$$(3.12) \quad \epsilon(\chi, \psi) = \chi(c) q^{a(\chi)/2} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(x/c) m'(U_F^{a(\chi)}).$$

Therefore the **modified** formula of epsilon factor of a character χ is:

$$(3.13) \quad \epsilon(\chi, \psi) = \chi(c) q^{-a(\chi)/2} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(x/c).$$

where $c = \pi_F^{a(\chi)+n(\psi)}$. Now if $u \in U_F$ is unit and replace $c = cu$, then we have

$$(3.14) \quad \epsilon(\chi, \psi, cu) = \chi(c) q^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x/u) \psi(x/cu) = \epsilon(\chi, \psi, c).$$

Therefore $\epsilon(\chi, \psi, c)$ **depends** only on the exponent $\nu_F(c) = a(\chi) + n(\psi)$. Therefore we can simply write $\epsilon(\chi, \psi, c) = \epsilon(\chi, \psi)$, because c is determined by $\nu_F(c) = a(\chi) + n(\psi)$ up to a unit u which has **no influence on** $\epsilon(\chi, \psi, c)$. If χ is unramified, i.e., $a(\chi) = 0$, therefore $\nu_F(c) = n(\psi)$. Then from the formula of $\epsilon(\chi, \psi, c)$, we can write

$$(3.15) \quad \epsilon(\chi, \psi, c) = \chi(c),$$

and therefore $\epsilon(1, \psi, c) = 1$ if $\chi = 1$ is the trivial character.

3.1. Some properties of $\epsilon(\chi, \psi)$.

(1) Let $b \in F^\times$ be the uniquely determined element such that $\psi' = b\psi$. Then

$$(3.16) \quad \epsilon(\chi, \psi', c') = \chi(b) \cdot \epsilon(\chi, \psi, c).$$

Proof. Here $\psi'(x) = (b\psi)(x) := \psi(bx)$ for all $x \in F$. It is an additive character of F . The existence and uniqueness of b is clear. From the definition of conductor of an additive character we have

$$n(\psi') = n(b\psi) = n(\psi) + \nu_F(b).$$

Here $c' \in F^\times$ is of valuation

$$\nu_F(c') = a(\chi) + n(\psi') = a(\chi) + \nu_F(b) + n(\psi) = \nu_F(b) + \nu_F(c) = \nu_F(bc).$$

Therefore $c' = bcu$ where $u \in U_F$ is some unit. Now

$$\begin{aligned} \epsilon(\chi, \psi', c') &= \epsilon(\chi, b\psi, bcu) \\ &= \epsilon(\chi, b\psi, bc) \\ &= \chi(bc) q^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) ((bc)^{-1}(b\psi))(x) \\ &= \chi(b) \cdot \chi(c) q^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(xc^{-1}) \\ &= \chi(b) \cdot \epsilon(\chi, \psi, c). \end{aligned}$$

□

(2) Let F/\mathbb{Q}_p be a local field inside $\overline{\mathbb{Q}_p}$. Let χ and ψ be a character of F^\times and F^+ respectively, and $c \in F^\times$ with valuation $\nu_F(c) = a(\chi) + n(\psi)$. If $\sigma \in \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is an automorphism, then:

$$\epsilon_F(\chi, \psi, c) = \epsilon_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)),$$

where $\chi^\sigma(y) := \chi(\sigma(y))$, $\psi^\sigma(y) := \psi(\sigma(y))$, for all $y \in \sigma^{-1}(F)$.

Proof. Let $L := \sigma^{-1}(F)$. Since σ is an automorphism of $\overline{\mathbb{Q}_p}$, then we have $O_F/P_F \cong O_L/P_L$, hence $q = q_L$, the cardinality of the residue field of L . We also can see that $a(\chi^\sigma) = a(\chi)$ and $n(\psi^\sigma) = n(\psi)$. Then from the formula of local constant we have

$$\begin{aligned} \epsilon_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)) &= \epsilon_L(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)) \\ &= \chi^\sigma(\sigma^{-1}(c)) q_L^{-\frac{a(\chi^\sigma)}{2}} \sum_{y \in \frac{U_L}{U_L^{a(\chi^\sigma)}}} (\chi^\sigma)^{-1}(y) \cdot ((\sigma^{-1}(c))^{-1} \psi^\sigma(y)) \\ &= \chi(c) q^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{x}{c}\right) \\ &= \epsilon_F(\chi, \psi, c). \end{aligned}$$

Here we put $y = \sigma^{-1}(x)$ and use $(\sigma^{-1}(c))^{-1} \psi^\sigma = (c^{-1} \psi)^\sigma$.

□

Remark 3.2. We can simply write as before $\epsilon_F(\chi, \psi) = \epsilon_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma)$. Tate in his paper [5] on local constants defines the local root number as:

$$\epsilon_F(\chi) := \epsilon_F(\chi, \psi_F) = \epsilon_F(\chi, \psi_F, d),$$

where ψ_F is the canonical character of F^\times and $d \in F^\times$ with $\nu_F(d) = a(\chi) + n(\psi_F)$. Therefore after fixing canonical additive character $\psi = \psi_F$, we can rewrite

$$\begin{aligned} \epsilon_F(\chi) &= \chi(d(\psi_F)), \text{ if } \chi \text{ is unramified,} \\ \epsilon(\chi) &= \epsilon_{\sigma^{-1}(F)}(\chi^\sigma). \end{aligned}$$

The last equality follows because the canonical character $\psi_{\sigma^{-1}(F)}$ is related to the canonical character ψ_F as: $\psi_{\sigma^{-1}(F)} = \psi_F^\sigma$.

(3) If χ is a multiplicative and ψ is a non-trivial additive character of F , then

$$\epsilon(\chi, \psi) \cdot \epsilon(\chi^{-1}, \psi) = \chi(-1).$$

Furthermore if the character $\chi : F^\times \rightarrow \mathbb{C}^\times$ is unitary (in particular if χ is of finite order), then

$$|\epsilon(\chi, \psi)|^2 = 1.$$

Proof. We prove these properties by using the modified formula (3.13) of local constant. We know that the additive characters are always unitary, hence

$$\psi(-x) = \psi(x)^{-1} = \overline{\psi}(x).$$

On the other hand we write $\psi(-x) = ((-1)\psi)(x)$, where $-1 \in F^\times$. Therefore $\overline{\psi} = (-1)\psi$. We also have $a(\chi) = a(\chi^{-1})$. Therefore by using equation (3.13) we have

$$\begin{aligned} \epsilon(\chi, \psi) \cdot \epsilon(\chi^{-1}, \psi) &= \chi(-1) \cdot q^{-a(\chi)} \sum_{x, y \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \chi(y) \psi\left(\frac{x-y}{c}\right) \\ &= \chi(-1) \cdot q^{-a(\chi)} \sum_{x, y \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{xy-y}{c}\right), \quad \text{replacing } x \text{ by } xy \\ &= \chi(-1) \cdot q^{-a(\chi)} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \varphi(x), \end{aligned}$$

where

$$(3.17) \quad \varphi(x) = \sum_{y \in \frac{U_F}{U_F^{a(\chi)}}} \psi\left(y \frac{x-1}{c}\right).$$

Therefore from equation (3.11) we have

$$\begin{aligned} \epsilon(\chi, \psi) \cdot \epsilon(\chi^{-1}, \psi) &= \chi(-1) \cdot q^{-a(\chi)} \cdot \{q^{a(\chi)} - q^{a(\chi)-1} \sum_{x \in \frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}} \chi^{-1}(x)\} \\ &= \chi(-1) - \chi(-1) \cdot q^{-1} \sum_{x \in \frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}} \chi^{-1}(x). \end{aligned}$$

Since the conductor of χ is $a(\chi)$, then it can be proved that $\sum_{x \in \frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}} \chi^{-1}(x) = 0$.

Thus we obtain

$$(3.18) \quad \epsilon(\chi, \psi) \cdot \epsilon(\chi^{-1}, \psi) = \chi(-1).$$

The right side of equation (3.18) is a sign, hence we may rewrite (3.18) as

$$\epsilon(\chi, \psi) \cdot \chi(-1) \epsilon(\chi^{-1}, \psi) = 1.$$

But we also know from our earlier property that

$$\chi(-1) \epsilon(\chi^{-1}, \psi) = \epsilon(\chi^{-1}, (-1)\psi) = \epsilon(\chi^{-1}, \overline{\psi}).$$

So the identity (3.18) rewrites as

$$\epsilon(\chi, \psi) \cdot \epsilon(\chi^{-1}, \overline{\psi}) = 1.$$

Now we assume that χ is unitary, hence

$$\epsilon(\chi^{-1}, \overline{\psi}) = \epsilon(\overline{\chi}, \overline{\psi}) = \overline{\epsilon(\chi, \psi)}$$

where the last equality is obvious from the formula (3.13). Now we see that for unitary χ the identity (3.18) rewrites as

$$|\epsilon(\chi, \psi)|^2 = 1.$$

□

Remark 3.3. From the functional equation (3.2), we can directly see the first part of the above property of local constant. Denote

$$(3.19) \quad \zeta(f, \chi) = \int f(x) \chi(x) d^\times x.$$

Now replacing f by \hat{f} in equation (3.19), and we get

$$(3.20) \quad \zeta(\hat{f}, \chi) = \int \hat{f}(x) \chi(x) d^\times x = \chi(-1) \cdot \zeta(f, \chi),$$

because dx is self-dual with respect to ψ , hence $\hat{f}(x) = f(-x)$ for all $x \in F^+$.

Again the functional equation (3.2) can be written as follows:

$$(3.21) \quad \zeta(\hat{f}, w_1 \chi^{-1}) = \epsilon(\chi, \psi, dx) \cdot \frac{L(w_1 \chi^{-1})}{L(\chi)} \cdot \zeta(f, \chi).$$

Now we replace f by \hat{f} , and χ by $w_1 \chi^{-1}$ in equation (3.21), and we obtain

$$(3.22) \quad \zeta(\hat{f}, \chi) = \epsilon(w_1 \chi^{-1}, \psi, dx) \cdot \frac{L(\chi)}{L(w_1 \chi^{-1})} \cdot \zeta(\hat{f}, w_1 \chi^{-1}).$$

Then by using equations (3.20), (3.21), from the above equation (3.22) we obtain:

$$(3.23) \quad \epsilon(\chi, \psi, dx) \cdot \epsilon(w_1 \chi^{-1}, \psi, dx) = \chi(-1).$$

Moreover, the convention $\epsilon(\chi, \psi)$ is actually as follows (cf. [6], p. 17, equation (3.6.4)):

$$\epsilon(\chi w_{s-\frac{1}{2}}, \psi) = \epsilon(\chi w_s, \psi, dx).$$

By using this relation from equation (3.23) we can conclude

$$\epsilon(\chi, \psi) \cdot \epsilon(\chi^{-1}, \psi) = \chi(-1).$$

(4) Known twisting formula of abelian epsilon factors:

(a) If χ_1 and χ_2 are two unramified characters of F^\times and ψ be a non-trivial additive character of F , then we have from equation (3.15)

$$(3.24) \quad \epsilon(\chi_1 \chi_2, \psi) = \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi).$$

(b) Let χ_1 be ramified and χ_2 unramified then (cf. [6], (3.2.6.3))

$$(3.25) \quad \epsilon(\chi_1 \chi_2, \psi) = \chi_2(\pi_F)^{a(\chi_1) + n(\psi)} \cdot \epsilon(\chi_1, \psi).$$

Proof. By the given condition $a(\chi_1) > a(\chi_2) = 0$. Therefore $a(\chi_1\chi_2) = a(\chi_1)$. Then we have

$$\begin{aligned}
\epsilon(\chi_1\chi_2, \psi) &= \chi_1\chi_2(c)q^{-a(\chi_1)/2} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} (\chi_1\chi_2)^{-1}(x)\psi(x/c) \\
&= \chi_1(c)\chi_2(c)q^{-a(\chi_1)/2} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi_1^{-1}(x)\chi_2^{-1}(x)\psi(x/c) \\
&= \chi_2(c)\chi_1(c)q^{-a(\chi_1)/2} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi_1^{-1}(x)\psi(x/c), \quad \text{since } \chi_2 \text{ unramified} \\
&= \chi_2(c)\epsilon(\chi_1, \psi) \\
&= \chi_2(\pi_F)^{a(\chi_1)+n(\psi)} \cdot \epsilon(\chi_1, \psi).
\end{aligned}$$

□

(c) We also have twisting formula of epsilon factor by Deligne (cf. [8], Lemma 4.16) under some special condition and which is as follows:

Let α and β be two multiplicative characters of a local field F such that $a(\alpha) \geq 2 \cdot a(\beta)$. Let ψ be an additive character of F . Let $y_{\alpha,\psi}$ be an element of F^\times such that

$$\alpha(1+x) = \psi(y_{\alpha,\psi}x)$$

for all $x \in F$ with valuation $\nu_F(x) \geq \frac{a(\alpha)}{2}$ (if $a(\alpha) = 0$, $y_{\alpha,\psi} = \pi_F^{-n(\psi)}$). Then

$$(3.26) \quad \epsilon(\alpha\beta, \psi) = \beta^{-1}(y_{\alpha,\psi}) \cdot \epsilon(\alpha, \psi).$$

3.2. Connection of different conventions for local constants. Mainly there are two conventions for local constants. They are due to Langlands ([9]) and Deligne ([8]). Recently Bushnell and Henniart ([1]) also give a convention of local constants. In this subsection we shall show the connection among all three conventions for local constants¹ We denote ϵ_{BH} as local constant of Bushnell-Henniart (introduced in Bushnell-Henniart, [1], Chapter 6).

On page 142 of [1], the authors define a rational function $\epsilon_{BH}(\chi, \psi, s) \in \mathbb{C}(q^{-s})$. From Theorem 23.5 on p.144 of [1] for ramified character $\chi \in \widehat{F}^\times$ and conductor² $n(\psi) = -1$ we have

$$(3.27) \quad \epsilon_{BH}(\chi, s, \psi) = q^{n(\frac{1}{2}-s)} \sum_{x \in \frac{U_F}{U_F^{n+1}}} \chi(\alpha x)^{-1} \psi(\alpha x) / q^{\frac{n+1}{2}},$$

where $n = a(\chi) - 1$, and $\alpha \in F^\times$ with $\nu_F(\alpha) = -n$.

¹The convention $\epsilon(\chi, \psi)$ is actually due to Langlands [9], and it is:

$$\epsilon_L(\chi, \psi, \frac{1}{2}) = \epsilon(\chi, \psi).$$

See equation (3.6.4) on p.17 of [6] for $V = \chi$.

²The definition of level of an additive character $\psi \in \widehat{F}$ in [1] on p.11 is the negative sign with our conductor $n(\psi)$, i.e., level of $\psi = -n(\psi)$.

Also from the Proposition 23.5 of [1] on p.143 for unramified character $\chi \in \widehat{F^\times}$ and $n(\psi) = -1$ we have

$$(3.28) \quad \epsilon_{BH}(\chi, s, \psi) = q^{s-\frac{1}{2}} \chi(\pi_F)^{-1}.$$

(1) **Connection between ϵ_{BH} and $\epsilon(\chi, \psi)$.**

$$\epsilon(\chi, \psi) = \epsilon_{BH}(\chi, \tfrac{1}{2}, \psi).$$

Proof. From [1], p.143, Lemma 1 we see:

$$\epsilon_{BH}(\chi, \tfrac{1}{2}, b\psi) = \chi(b) \epsilon_{BH}(\chi, \tfrac{1}{2}, \psi)$$

for any $b \in F^\times$. But we have seen already that $\epsilon(\chi, b\psi) = \chi(b) \cdot \epsilon(\chi, \psi)$ has the same transformation rule. If we fix one nontrivial ψ then all other nontrivial ψ' are uniquely given as $\psi' = b\psi$ for some $b \in F^\times$. Because of the parallel transformation rules it is now enough to verify our assertion for a single ψ . Now we take $\psi \in \widehat{F^+}$ with $n(\psi) = -1$, hence $\nu_F(c) = a(\chi) - 1$. Then we obtain

$$\epsilon(\chi, \psi) = \epsilon(\chi, \psi, c) = \chi(c) q^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(c^{-1}x).$$

We compare this to the equation (3.27). There the notation is $n = a(\chi) - 1$ and the assumption is $n \geq 0$. This means we have $\nu_F(c) = n$, hence we may take $\alpha = c^{-1}$ and then comparing our formula with equation (3.27), we see that

$$\epsilon(\chi, \psi) = \epsilon_{BH}(\chi, \tfrac{1}{2}, \psi)$$

in the case when $n(\psi) = -1$.

We are still left to prove our assertion if χ is unramified, i.e., $a(\chi) = 0$. Again we can reduce to the case where $n(\psi) = -1$. Then our assertion follows from equation 3.28. \square

Remark 3.4. From Corollary 23.4.2 of [1], on p.142, for $s \in \mathbb{C}$, we can write

$$(3.29) \quad \epsilon_{BH}(\chi, s, \psi) = q^{(\frac{1}{2}-s)n(\chi, \psi)} \cdot \epsilon_{BH}(\chi, \tfrac{1}{2}, \psi),$$

for some $n(\chi, \psi) \in \mathbb{Z}$. In fact here $n(\chi, \psi) = a(\chi) + n(\psi)$. From above connection, we just see $\epsilon(\chi, \psi) = \epsilon_{BH}(\chi, \tfrac{1}{2}, \psi)$. Thus for arbitrary $s \in \mathbb{C}$, we obtain

$$(3.30) \quad \epsilon_{BH}(\chi, s, \psi) = q^{(\frac{1}{2}-s)(a(\chi)+n(\psi))} \cdot \epsilon(\chi, \psi).$$

This equation (3.30) is very important for us. We shall use this to connect with Deligne's convention.

In [6] there is defined a number $\epsilon_D(\chi, \psi, dx)$ depending on χ, ψ and a Haar measure dx on F . This convention is due to Deligne [8]. We write ϵ_D for Deligne's convention in order to distinguish it from the $\epsilon_{BH}(\chi, \tfrac{1}{2}, \psi)$ introduced in Bushnell-Henniart [1].

In the next Lemma we give the connection between Bushnell-Henniart and Deligne conventions for local constants.

(2) **The connection between ϵ_D and ϵ_{BH} :**

Lemma 3.5. *We have the relation*

$$\epsilon_{BH}(\chi, s, \psi) = \epsilon_D(\chi \cdot \omega_s, \psi, dx_\psi),$$

where $\omega_s(x) = |x|_F^s = q^{-s\nu_F(x)}$ is unramified character of F^\times corresponding to complex number s , and where dx_ψ is the self-dual Haar measure corresponding to the additive character ψ .

Proof. From equation (3.6.4) of [6], we know that

$$(3.31) \quad \epsilon_L(\chi, s, \psi) := \epsilon_L(\chi\omega_{s-\frac{1}{2}}, \psi) = \epsilon_D(\chi\omega_s, \psi, dx_\psi).$$

We prove this connection by using the relations (3.30) and (3.31). From equation (3.31) we can write our $\epsilon(\chi, \psi) = \epsilon_D(\chi\omega_{\frac{1}{2}}, \psi, dx_\psi)$. Therefore when $s = \frac{1}{2}$, we have the relation:

$$(3.32) \quad \epsilon_{BH}(\chi, \frac{1}{2}, \psi) = \epsilon_D(\chi\omega_{\frac{1}{2}}, \psi, dx_\psi),$$

since $\epsilon(\chi, \psi) = \epsilon_{BH}(\chi, \frac{1}{2}, \psi)$.

We know that $\omega_s(x) = q^{-s\nu_F(x)}$ is an unramified character of F^\times . So when χ is also unramified, we can write

$$(3.33) \quad \epsilon(\chi\omega_{s-\frac{1}{2}}, \psi) = \omega_{s-\frac{1}{2}}(c) \cdot \chi(c) = q^{(\frac{1}{2}-s)n(\psi)} \epsilon_{BH}(\chi, \frac{1}{2}, \psi) = \epsilon_{BH}(\chi, s, \psi).$$

And when χ is ramified character, i.e., conductor $a(\chi) > 0$, from Tate's theorem for unramified twist (see property 2.3.1(4b)), we can write

$$\begin{aligned} \epsilon(\chi\omega_{s-\frac{1}{2}}, \psi) &= \omega_{s-\frac{1}{2}}(\pi_F^{a(\chi)+n(\psi)}) \cdot \epsilon(\chi, \psi) \\ &= q^{-(s-\frac{1}{2})(a(\chi)+n(\psi))} \cdot \epsilon(\chi, \psi) \\ &= q^{(\frac{1}{2}-s)(a(\chi)+n(\psi))} \cdot \epsilon_{BH}(\chi, \frac{1}{2}, \psi) \\ &= \epsilon_{BH}(\chi, s, \psi). \end{aligned}$$

Furthermore from equation (3.31), we have

$$(3.34) \quad \epsilon(\chi\omega_{s-\frac{1}{2}}, \psi) = \epsilon_D(\chi\omega_s, \psi, dx_\psi).$$

Therefore finally we can write

$$(3.35) \quad \epsilon_{BH}(\chi, s, \psi) = \epsilon_D(\chi\omega_s, \psi, dx_\psi).$$

□

Corollary 3.6. *For our W we have :*

$$\begin{aligned} \epsilon(\chi, \psi) &= \epsilon_{BH}(\chi, \frac{1}{2}, \psi) = \epsilon_D(\chi\omega_{\frac{1}{2}}, \psi, dx_\psi) \\ \epsilon(\chi\omega_{s-\frac{1}{2}}, \psi) &= \epsilon_{BH}(\chi, s, \psi) = \epsilon_D(\chi\omega_s, \psi, dx_\psi). \end{aligned}$$

Proof. From the equations (3.6.1) and (3.6.4) of [6] for χ and above two connections the assertions follow. □

3.3. Local constants for virtual representations.

- (1) To extend the concept of local constant, we need go from 1-dimensional to other virtual representations ρ of the Weil groups W_F of nonarchimedean local field F . According to Tate [5], the root number $\epsilon(\chi) := \epsilon(\chi, \psi_F)$ extends to $\epsilon(\rho)$, where ψ_F is the canonical additive character of F . More generally, $\epsilon(\chi, \psi)$ extends to $\epsilon(\rho, \psi)$, and if E/F is a finite separable extension then one has to take $\psi_E = \psi_F \circ \text{Tr}_{E/F}$ for the extension field E .

According to Bushnell-Henniart [1], Theorem on p.189, the functions $\epsilon_{BH}(\chi, s, \psi)$ extend to $\epsilon_{BH}(\rho, s, \psi_E)$, where $\psi_E = \psi \circ \text{Tr}_{E/F}$ ³. According to Tate [6], Theorem (3.4.1) the functions $\epsilon_D(\chi, \psi, dx)$ extends to $\epsilon_D(\rho, \psi, dx)$. In order to get **weak inductivity** (cf. [5]) we have again to use $\psi_E = \psi \circ \text{Tr}_{E/F}$ if we consider extensions. Then according to Tate [6] (3.6.4) the Corollary 3.6 turns into

Corollary 3.7. *For the virtual representations of the Weil groups we have*

$$\begin{aligned} \epsilon(\rho\omega_{E,s-\frac{1}{2}}, \psi_E) &= \epsilon_{BH}(\rho, s, \psi_E) = \epsilon_D(\rho\omega_{E,s}, \psi_E, dx_{\psi_E}). \\ \epsilon(\rho, \psi_E) &= \epsilon_{BH}(\rho, \tfrac{1}{2}, \psi_E) = \epsilon_D(\rho\omega_{E,\frac{1}{2}}, \psi_E, dx_{\psi_E}). \end{aligned}$$

Note that on the level of field extension E/F we have to use $\omega_{E,s}$ which is defined as

$$\omega_{E,s}(x) = |x|_E^s = q_E^{-s\nu_E(x)}.$$

We also know that $q_E = q^{f_{E/F}}$ and $\nu_E = \frac{1}{f_{E/F}} \cdot \nu_F(N_{E/F})$ (cf. [3], p.41, Theorem 2.5), therefore we can easily see that

$$\omega_{E,s} = \omega_{F,s} \circ N_{E/F}.$$

Since the norm map $N_{E/F} : E^\times \rightarrow F^\times$ corresponds via class field theory to the injection map $G_E \hookrightarrow G_F$, Tate [6] beginning from (1.4.6), simply writes $\omega_s = ||^s$ and consider ω_s as an unramified character of the Galois group (or of the Weil group) instead as a character on the field. Then Corollary 3.7 turns into

$$(3.36) \quad \epsilon(\rho\omega_{s-\frac{1}{2}}, \psi_E) = \epsilon_{BH}(\rho, s, \psi_E) = \epsilon_D(\rho\omega_s, \psi_E, dx_{\psi_E}),$$

for all field extensions, where ω_s is to be considered as 1-dimensional representation of the Weil group $W_E \subset G_E$ if we are on the E -level. The left side equation (3.36) is the ϵ -factor of Langlands (see [6], (3.6.4)).

- (2) The functional equation (3.18) extends to

$$(3.37) \quad \epsilon(\rho, \psi) \cdot \epsilon(\rho^V, \psi) = \det_\rho(-1),$$

where ρ is any virtual representation of the Weil group W_F , ρ^V is the contragredient and ψ is any nontrivial additive character of F . This is formula (3) on p.190 of [1] for $s = \frac{1}{2}$.

³ Note that they fix a base field F and a nontrivial $\psi = \psi_F$ (which not to be the canonical character used in Tate [5]) but then if E/F is an extension they always use $\psi_E = \psi \circ \text{Tr}_{E/F}$.

- (3) Moreover the transformation law (3.4.5) of [6] can (on the F -level) be written as:
unramified character twist

$$(3.38) \quad \epsilon_D(\rho\omega_s, \psi, dx) = \epsilon_D(\rho, \psi, dx) \cdot \omega_{F,s}(c_{\rho,\psi})$$

for any $c = c_{\rho,\psi}$ such that $\nu_F(c) = a(\rho) + n(\psi)\dim(\rho)$. It implies that also for the root number on the F -level we have

$$(3.39) \quad \epsilon(\rho\omega_s, \psi) = \epsilon(\rho, \psi) \cdot \omega_{F,s}(c_{\rho,\psi}).$$

4. Generalized twisting formula of epsilon factors

4.1. Local Gauss sum. Let m be a nonzero positive integer. Let χ be a nontrivial multiplicative character of F with conductor $a(\chi)$ and $\psi : F \rightarrow \mathbb{C}^\times$ be an additive character of F with conductor $n(\psi)$. We define the local character sum of a character χ :

$$(4.1) \quad G(\chi, \psi, m) = \sum_{x \in \frac{U_F}{U_F^m}} \chi^{-1}(x) \psi(x/c),$$

where $c = \pi_F^{a(\chi)+n(\psi)}$. When $m = a(\chi)$, we call $G(\chi, \psi, a(\chi))$ as the **local Gauss sum** of character χ .

Proposition 4.1. *The definition of local Gauss sum $G(\chi, \psi, a(\chi))$ does not depend on the choice of the coset representatives of $U_F \bmod U_F^{a(\chi)}$.*

Proof. It is very easy to see from the definition of local Gauss sum. If we change one of the coset representatives x to xu where $u \in U_F^{a(\chi)}$ in $G(\chi, \psi)$ and we have

$$\begin{aligned} G(\chi, \psi, a(\chi)) &= \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(x/c) \\ &= \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(xu) \psi(xu/c) \quad \text{replacing } x \text{ by } xu, u \in U_F^{a(\chi)} \\ &= \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(x/c) \psi\left(\frac{x}{c}(u-1)\right) \\ &= \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(x/c). \end{aligned}$$

Since $P_F^{-n(\psi)}$ is a fractional ideal of O_F , then $\frac{x}{c}(u-1) \in P_F^{-n(\psi)}$ for $x \in U_F$. Therefore $\psi(\frac{x}{c}(u-1)) = 1$ for all $x \in U_F$ and $u \in U_F^{a(\chi)}$. This proves the local character sum is independent of the choice of coset representatives of $U_F \bmod U_F^{a(\chi)}$. Therefore definition of local character sum does not depend on the choice of coset representatives of x . \square

In the next proposition we compute the absolute value of $G(\chi, \psi, a(\chi))$ by using Lemma 3.1.

Proposition 4.2. *If χ is a ramified character of F^\times , then*

$$|G(\chi, \psi, a(\chi))| = q^{\frac{a(\chi)}{2}}.$$

Proof. We can write

$$(4.2) \quad \int_{U_F} \chi^{-1}(x) \psi(x/c) dx = \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(x/c) \times m'(U_F^{a(\chi)}).$$

where $c = \pi_F^{a(\chi)+n(\psi)}$ and m' is the Haar measure which is normalized so that $m'(O_F) = 1$. Now from equation (4.2) we have

$$\begin{aligned} \left| \int_{U_F} \chi^{-1}(x) \psi(x/c) dx \right| &= \left| \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi(x/c) \right| \times |m'(U_F^{a(\chi)})| \\ &= |G(\chi, \psi, a(\chi))| q^{-a(\chi)}, \quad \text{since } m'(U_F^{a(\chi)}) = q^{-a(\chi)}. \end{aligned}$$

Therefore from Lemma 3.1 we have

$$(4.3) \quad |G(\chi, \psi, a(\chi))| = q^{\frac{a(\chi)}{2}}.$$

□

Furthermore, from Lemma 3.1 it can be proved that

$$(4.4) \quad \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{x}{\pi_F^{l+m}}\right) = 0,$$

for non-zero integers $m \neq 0$ and $l = a(\chi) + n(\psi)$.

In the next lemma, we see the relation between two sums $G(\chi, \psi, n_1)$, $G(\chi, \psi, n_2)$, where $n_1 > n_2$ and here we mean

$$G(\chi, \psi, n_i) = \sum_{x \in \frac{U_F}{U_F^{n_i}}} \chi^{-1}(x) \psi(x/c), \quad i = 1, 2$$

where $c = \pi_F^{a(\chi)+n(\psi)}$.

Lemma 4.3.

$$(4.5) \quad G(\chi, \psi, n_1) = q^m G(\chi, \psi, n_2).$$

where $m = n_1 - n_2$.

Proof. From Lemma 3.1, it is straight forward. We have

$$\begin{aligned} \int_{U_F} \chi^{-1}(x) \psi\left(\frac{x}{c}\right) dx &= \sum_{x \in \frac{U_F}{U_F^{n_1}}} \chi^{-1}(x) \psi\left(\frac{x}{c}\right) \times m'(U_F^{n_1}) \\ &= q^{-n_1} \sum_{x \in \frac{U_F}{U_F^{n_1}}} \chi^{-1}(x) \psi\left(\frac{x}{c}\right) \quad \text{since } m'(U_F^{n_1}) = q^{-n_1} \\ &= q^{-n_1} G(\chi, \psi, n_1). \end{aligned}$$

Similarly, we can express

$$\int_{U_F} \chi^{-1}(x) \psi\left(\frac{x}{c}\right) dx = q^{-n_2} G(\chi, \psi, n_2).$$

Comparing these two above equations we obtain

$$G(\chi, \psi, n_1) = q^{n_1-n_2} G(\chi, \psi, n_2) = q^m G(\chi, \psi, n_2).$$

□

4.2. Local Jacobi sum. Let χ_1 and χ_2 be two nontrivial characters of F^\times . For any $t \in \frac{U_F}{U_F^n}$, where $n \geq 1$, we define the local Jacobi sum for characters χ_1 and χ_2 :

$$(4.6) \quad J_t(\chi_1, \chi_2, n) = \sum_{\substack{x \in \frac{U_F}{U_F^n} \\ t-x \in U_F}} \chi_1^{-1}(x) \chi_2^{-1}(t-x).$$

Proposition 4.4.

$$(4.7) \quad J_1(\chi_1, \chi_2, n) = \chi_1 \chi_2(t) \cdot J_t(\chi_1, \chi_2, n), \quad \text{for any } t \in \frac{U_F}{U_F^n}.$$

Proof. For any $t \in \frac{U_F}{U_F^n}$, from the definition of Jacobi sum, we have

$$\begin{aligned} J_t(\chi_1, \chi_2, n) &= \sum_{\substack{x \in \frac{U_F}{U_F^n} \\ t-x \in U_F}} \chi_1^{-1}(x) \chi_2^{-1}(t-x) \\ &= \sum_{\substack{x/t \in \frac{U_F}{U_F^n} \\ 1-x/t \in U_F}} (\chi_1 \chi_2)^{-1}(t) \chi_1^{-1}(x/t) \chi_2^{-1}(1-x/t) \\ &= (\chi_1 \chi_2)^{-1}(t) \sum_{\substack{s=x/t \in \frac{U_F}{U_F^n} \\ 1-s \in U_F}} \chi_1^{-1}(s) \chi_2^{-1}(1-s) \\ &= (\chi_1 \chi_2)^{-1}(t) J_1(\chi_1, \chi_2, n). \end{aligned}$$

Therefore

$$J_1(\chi_1, \chi_2, n) = \chi_1 \chi_2(t) \cdot J_t(\chi_1, \chi_2, n), \quad \text{for any } t \in \frac{U_F}{U_F^n}.$$

□

In the following theorem, we give the twisting formula of epsilon factor via this local Jacobi sums.

Theorem 4.5. Let χ_1 and χ_2 be two ramified characters of F^\times with conductors n and m respectively. Let r be the conductor of character $\chi_1 \chi_2$. Then

$$(4.8) \quad \epsilon(\chi_1 \chi_2, \psi) = \begin{cases} \frac{q^{\frac{n}{2}} \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = m = r, \\ \frac{q^{\frac{r}{2}} \chi_1 \chi_2(\pi_F^{r-n}) \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = m > r, \\ \frac{q^{n-\frac{m}{2}} \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)} & \text{when } n = r > m, \end{cases}$$

Proof. We know the formula of epsilon of a character χ of F^\times and it is:

$$(4.9) \quad \epsilon(\chi, \psi) = \chi(c)q^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(x/c) = \chi(c)q^{-\frac{a(\chi)}{2}} G(\chi, \psi, a(\chi)),$$

where $c = \pi_F^{a(\chi)+n(\psi)}$

Now we divide this proof into three cases.

Case-1 when $n = m = r$. By the definition of epsilon factor, in this case we have

$$(4.10) \quad \epsilon(\chi_1, \psi) = \chi_1(c_1)q^{-\frac{n}{2}} \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x)\psi(x/c_1),$$

and

$$(4.11) \quad \epsilon(\chi_2, \psi) = \chi_2(c_2)q^{-\frac{m}{2}} \sum_{y \in \frac{U_F}{U_F^m}} \chi_2^{-1}(y)\psi(y/c_2),$$

where $c_1 = \pi_F^{n+n(\psi)}$ and $c_2 = \pi_F^{m+n(\psi)}$. Since $n = m$, then we consider $c = c_1 = c_2$.

Now from equation (4.10) and (4.11) we have

$$\begin{aligned} \epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi) &= q^{-n}\chi_1\chi_2(c) \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x)\psi(x/c) \times \sum_{y \in \frac{U_F}{U_F^n}} \chi_2^{-1}(y)\psi(y/c) \\ &= q^{-n}\chi_1\chi_2(c) \sum_{x, y \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x)\chi_2^{-1}(y)\psi(x/c)\psi(y/c) \\ &= q^{-n}\chi_1\chi_2(c) \sum_{x, y \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x)\chi_2^{-1}(y)\psi\left(\frac{x+y}{c}\right) \\ &= q^{-n}\chi_1\chi_2(c) \sum_{\substack{x \in \frac{U_F}{U_F^n} \\ t-x \in \frac{U_F}{U_F^n}}} \chi_1^{-1}(x)\chi_2^{-1}(t-x)\psi\left(\frac{t}{c}\right) \\ &= q^{-n}\chi_1\chi_2(c) \sum_{a=0}^{n-1} \left\{ \sum_{u \in \frac{U_F}{U_F^{n-a}}} \left\{ \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}\left(\frac{x}{\pi_F^a u}\right)\chi_2^{-1}\left(1 - \frac{x}{\pi_F^a u}\right) \right\} (\chi_1\chi_2)^{-1}(\pi_F^a u)\psi\left(\frac{\pi_F^a u}{c}\right) \right\} \\ &= q^{-n}\chi_1\chi_2(c) \sum_{a=0}^{n-1} \left\{ \sum_{u \in \frac{U_F}{U_F^{n-a}}} \left\{ \sum_{s=x/u \in \frac{U_F}{U_F^{n-a}}} \chi_1^{-1}\left(\frac{s}{\pi_F^a}\right)\chi_2^{-1}\left(1 - \frac{s}{\pi_F^a}\right) \right\} (\chi_1\chi_2)^{-1}(\pi_F^a u)\psi\left(\frac{\pi_F^a u}{c}\right) \right\} \\ &= q^{-n}\chi_1\chi_2(c) \sum_{a=0}^{n-1} \left\{ \sum_{s \in \frac{U_F}{U_F^{n-a}}} \chi_1^{-1}\left(\frac{s}{\pi_F^a}\right)\chi_2^{-1}\left(1 - \frac{s}{\pi_F^a}\right) \times \sum_{u \in \frac{U_F}{U_F^{n-a}}} (\chi_1\chi_2)^{-1}(\pi_F^a u)\psi\left(\frac{\pi_F^a u}{c}\right) \right\} \\ (4.12) \quad &= q^{-n}\chi_1\chi_2(c) \sum_{a=0}^{n-1} \{J'_1(\chi_1, \chi_2, n) \times G'(\chi_1\chi_2, \psi)\} \end{aligned}$$

where

$$(4.13) \quad G'(\chi_1 \chi_2, \psi) = \sum_{u \in \frac{U_F}{U_F^{n-a}}} (\chi_1 \chi_2)^{-1}(\pi_F^a u) \psi\left(\frac{\pi_F^a u}{c}\right),$$

and

$$(4.14) \quad J'_1(\chi_1, \chi_2, n) = \sum_{s \in \frac{U_F}{U_F^{n-a}}} \chi_1^{-1}\left(\frac{s}{\pi_F^a}\right) \chi_2^{-1}\left(1 - \frac{s}{\pi_F^a}\right)$$

In the above equations (4.13) and (4.14), we assume $t = x + y$ where both x and y are in $\frac{U_F}{U_F^n}$ and this t can be written as $t = \pi_F^a u$, where a varies over $\{0, 1, \dots, n-1\}$ and $u \in \frac{U_F}{U_F^{n-a}}$. Furthermore, by using Lemma 3.1, for $a \neq 0$, we have

$$\sum_{u \in \frac{U_F}{U_F^{n-a}}} (\chi_1 \chi_2)^{-1}(u) \psi\left(\frac{\pi_F^a u}{c}\right) = 0.$$

Therefore, for $a \neq 0$, we can write

$$\begin{aligned} G'(\chi_1 \chi_2, \psi) &= \sum_{u \in \frac{U_F}{U_F^{n-a}}} (\chi_1 \chi_2)^{-1}(\pi_F^a u) \psi\left(\frac{\pi_F^a u}{c}\right) \\ &= \chi_1 \chi_2(\pi_F^{-a}) \sum_{u \in \frac{U_F}{U_F^{n-a}}} (\chi_1 \chi_2)^{-1}(u) \psi\left(\frac{\pi_F^a u}{c}\right) \\ &= 0. \end{aligned}$$

Therefore, we have to take $a = 0$, because left side of equation (4.12) is **nonzero**, therefore $t = u \in \frac{U_F}{U_F^n}$. Now, put $a = 0$ in equation (4.13), then we have from equation (4.12)

$$\begin{aligned} \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi) &= q^{-n} \chi_1 \chi_2(c) J'_1(\chi_1 \chi_2, n) \sum_{\alpha \in \frac{U_F}{U_F^n}} (\chi_1 \chi_2)^{-1}(\alpha) \psi\left(\frac{\alpha}{c}\right) \\ &= q^{-n} \chi_1 \chi_2(c) J_1(\chi_1, \chi_2, n) G(\chi_1 \chi_2, \psi, n) \\ &= q^{-\frac{n}{2}} J_1(\chi_1, \chi_2, n) \epsilon(\chi_1 \chi_2, \psi). \end{aligned}$$

Therefore, in this case we have

$$(4.15) \quad \epsilon(\chi_1 \chi_2, \psi) = \frac{q^{\frac{n}{2}} \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)}.$$

Case-2 When $n = m > r$. Like case-1 in this case, it can be showed that $t = x + y \in \frac{U_F}{U_F^n}$ when $x, y \in \frac{U_F}{U_F^n}$. Since $c_1 = c_2$, then let $c = c_1 = c_2$. In this situation we have:

$$\begin{aligned}
\epsilon(\chi_1, \psi)\epsilon(\chi_2, \psi) &= q^{-n}\chi_1\chi_2(c) \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x)\psi(x/c) \times \sum_{y \in \frac{U_F}{U_F^n}} \chi_2^{-1}(y)\psi(y/c) \\
&= q^{-n}\chi_1\chi_2(c) \sum_{x, y \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x)\chi_2^{-1}(y)\psi(x/c)\psi(y/c) \\
&= q^{-n}\chi_1\chi_2(c) \sum_{x, y \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x)\chi_2^{-1}(y)\psi\left(\frac{x+y}{c}\right) \\
&= q^{-n}\chi_1\chi_2(c) \sum_{t, x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x)\chi_2^{-1}(t-x)\psi\left(\frac{t}{c}\right) \\
&= q^{-n}\chi_1\chi_2(c) \sum_{t \in \frac{U_F}{U_F^n}} \left\{ \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x/t)\chi_2^{-1}(1-x/t) \right\} (\chi_1\chi_2)^{-1}(t)\psi\left(\frac{t}{c}\right) \\
&= q^{-n}\chi_1\chi_2(c) \sum_{s=x/t \in \frac{U_F}{U_F^n}} \chi_1^{-1}(s)\chi_2^{-1}(1-s) \times \sum_{t \in \frac{U_F}{U_F^n}} (\chi_1\chi_2)^{-1}(t)\psi\left(\frac{t}{c}\right) \\
&= q^{-n}\chi_1\chi_2(c) J_1(\chi_1, \chi_2, n) \times G(\chi_1\chi_2, \psi, n) \\
&= q^{-n}\chi_1\chi_2(c) J_1(\chi_1, \chi_2, n) \times q^{n-r} G(\chi_1\chi_2, \psi, r) \quad \text{using Lemma 4.3} \\
&= q^{-\frac{r}{2}} \chi_1\chi_2(\pi_F^{n-r}) J_1(\chi_1, \chi_2, n) \chi_1\chi_2(\pi_F^r) q^{-\frac{r}{2}} G(\chi_1\chi_2, \psi, r) \\
&= q^{-\frac{r}{2}} \chi_1\chi_2(\pi_F^{n-r}) J_1(\chi_1, \chi_2, n) \epsilon(\chi_1\chi_2, \psi), \quad \text{since } a(\chi_1\chi_2) = r.
\end{aligned}$$

Therefore, in this condition we have the twisting formula of epsilon factor:

$$(4.16) \quad \epsilon(\chi_1\chi_2, \psi) = \frac{q^{\frac{r}{2}} \chi_1\chi_2(\pi_F^{r-n}) \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)}.$$

Case-3 when $n = r > m$. If conductor $a(\chi_1) > a(\chi_2)$, then conductor $a(\chi_1\chi_2) = \max(a(\chi_1), a(\chi_2)) = a(\chi_1)$. Therefore, we are in this situation: $n = r > m$. In this case

c_1 can be written as $c_1 = c_2 \pi_F^{n-m}$. If $x, z \in \frac{U_F}{U_F^n}$, then $x + \pi_F^{n-m} z \in \frac{U_F}{U_F^n}$.

$$\begin{aligned}
\epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi) &= q^{-\frac{m+n}{2}} \chi_1(c_1) \chi_2(c_2) \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x) \psi(x/c_1) \times \sum_{z \in \frac{U_F}{U_F^n}} \chi_2^{-1}(z) \psi(z/c_2) \\
&= q^{-\frac{m+n}{2}} \chi_2(\pi_F^{m-n}) \chi_1 \chi_2(c_1) \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x) \psi(x/c_1) \times \sum_{z \in \frac{U_F}{U_F^n}} \chi_2^{-1}(z) \psi\left(\frac{z \pi_F^{n-m}}{c_1}\right) \\
&= q^{-\frac{m+n}{2}} \chi_2(\pi_F^{m-n}) \chi_1 \chi_2(c_1) \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x) \psi(x/c_1) \times q^{m-n} \sum_{z \in \frac{U_F}{U_F^n}} \chi_2^{-1}(z) \psi\left(\frac{z \pi_F^{n-m}}{c_1}\right) \\
&= q^{\frac{m}{2} - \frac{3n}{2}} \chi_2(\pi_F^{m-n}) \chi_1 \chi_2(c_1) \sum_{x, z \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x) \chi_2^{-1}(z) \psi(x/c_1) \psi\left(\frac{z \pi_F^{n-m}}{c_1}\right) \\
&= q^{\frac{m}{2} - \frac{3n}{2}} \chi_2(\pi_F^{m-n}) \chi_1 \chi_2(c_1) \sum_{x, z \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x) \chi_2^{-1}(z) \psi\left(\frac{x + z \pi_F^{n-m}}{c_1}\right) \\
&= q^{\frac{m}{2} - \frac{3n}{2}} \chi_2(\pi_F^{m-n}) \chi_1 \chi_2(c_1) \chi_2(\pi_F^{n-m}) \sum_{x, t \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x) \chi_2^{-1}(t-x) \psi\left(\frac{t}{c_1}\right) \\
&= q^{\frac{m}{2} - \frac{3n}{2}} \chi_1 \chi_2(c_1) \sum_{t \in \frac{U_F}{U_F^n}} \left\{ \sum_{x \in \frac{U_F}{U_F^n}} \chi_1^{-1}(x/t) \chi_2^{-1}(1-x/t) \right\} (\chi_1 \chi_2)^{-1}(t) \psi\left(\frac{t}{c_1}\right) \\
&= q^{\frac{m}{2} - \frac{3n}{2}} \chi_1 \chi_2(c_1) \sum_{s=x/t \in \frac{U_F}{U_F^n}} \chi_1^{-1}(s) \chi_2^{-1}(1-s) \times \sum_{t \in \frac{U_F}{U_F^n}} (\chi_1 \chi_2)^{-1}(t) \psi\left(\frac{t}{c_1}\right) \\
&= q^{\frac{m}{2} - \frac{3n}{2}} \chi_1 \chi_2(c_1) J_1(\chi_1, \chi_2, n) \times \sum_{t \in \frac{U_F}{U_F^n}} (\chi_1 \chi_2)^{-1}(t) \psi\left(\frac{t}{c_1}\right) \\
&= q^{\frac{m}{2} - n} J_1(\chi_1, \chi_2, n) \chi_1 \chi_2(c_1) q^{-\frac{n}{2}} \times G(\chi_1 \chi_2, \psi, n) \\
&= q^{\frac{m}{2} - n} J_1(\chi_1, \chi_2, n) \epsilon(\chi_1 \chi_2, \psi).
\end{aligned}$$

Therefore we have the formula:

$$(4.17) \quad \epsilon(\chi_1 \chi_2, \psi) = \frac{q^{n-\frac{m}{2}} \epsilon(\chi_1, \psi) \epsilon(\chi_2, \psi)}{J_1(\chi_1, \chi_2, n)}.$$

□

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